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The Return of the Abacus in Teaching Subtraction? Correcting the Almost-Correct

A child's subtraction error is noted on page 71 of the text *Mathematics for Elementary School Teachers* (Bassarear, 5th ed.). The problem is 52 —16, and the child's answer is 44. The question to address is why did the child make this error? Where did the child get this algorithm? Unless we know the algorithm used and can "undo" it, learning an accurate algorithm is not likely. Further, if not identified and corrected, this will be the algorithm sustained by the child into adulthood and into later math classes. And, given this error, there may be a legitimate expectation that the child has almost-correct algorithms for other arithmetical and algebraic operations.

So what about this specific error-producing algorithm? In essence, we taught this algorithm to the child, and it was learned well! There is a tendency for teachers to describe subtraction with single-digit problems as "the difference between two numbers." The child then applies this algorithm to multiple-digit subtraction problems by parsing each place-value column into a unique single-digit subtraction, allowing the child to use the "difference" algorithm. This child's misconception may have been avoided had the early work with subtraction been done with an abacus. Take a look.

This child's error is related to a sophisticated subtraction algorithm, which incidentally doesn't involve borrowing. Borrowing unto itself is a somewhat bothersome procedure, notably in problems like 10020003 – 1234567. But take a look at how this child's error is linked to the abacus, other subtraction algorithms, and computer subtraction.

Here is how. As an example, 24 —7 on an abacus is "subtract 10 and add 3," which symbolically is —10 + 3, or algebraically —7. So, combining this with the numbers from the problems, the —7 is replaced with —10 + 3, and the problem <u>in the units column</u> is now 4 —10 + 3. Technically, the —10 is not allowable in the unit's column, so the values in the unit's column are 0 + 4 + 3, the correct answer being 7. The —1, which is (technically) in the tens column, gets added to the 2, giving 2 —1 or 1, the final answer being 17. How does this connect to the 10s and 9s complement methods?

For the 10s complement, in this case, and every case, the number in the subtrahend which gets added is really the original subtrahend value subtracted from 10. Other examples: 23 – 8 gives 3 – 10 + 2 = 5; 23 – 4 gives 3 — 10 + 6 = 9, etc. so this algorithm always works. Recall that the —1 gets operated on in its proper place-value column. So, basically, each value in the subtrahend is replaced with its 10s complement (which is found by subtracting it from 10, or arithmetically 10 − n), and every place-value unit except the units column has a −1 "carried" to it. If the −1 were shown as part of the subtraction process, the problem 12345 – 5169 would look like this (spaces added for clarification):

		+1			
	−1	−1	−1	−1	
	1	2	3	4	5
+		5	9	4	1
		7	1	7	6

This is the correct answer; but writing it down shows it to be a correct, but somewhat cumbersome, algorithm. Notice the +1 in the thousands column, which comes from addition in the hundreds column. Also notice that there was no number in the 10-thousands column, so there was also no 10s complement placed there. It could be argued that there really is a "zero" in that place but that we do not typically write leading zeros. However, if the zero were allowed, the value in the 10-thousands column would still be zero (again, a leading zero in the number 07176) because the 10s complement of zero is 10, leaving the 10-thousands column −1 + 1, or zero. Further, the 1 in the "10" in the 10-thousands column goes to the 100-thousands column; and if this is combined with a −1, since every place-value unit except the units column has a −1 "carried" to it, it zeros out for the 100-thousands column. Again, this is an accurate answer but cumbersome algorithm.

This "cumbersomeness" can be eliminated by showing how the above problem, based on the 10s complement, provides a ready transition to a 9s complement method. The 9s complement of a number is that number subtracted from 9 (9 − n, just as 10 − n is the 10s complement). So, the 9s complement of 5 is 4, of 1 is 8, of 3 is 6, etc., and it seems easier for children to find a 9s complement than to find a 10s complement. In a most technical concern, the 10s complement method requires children to know how to "borrow," even if it is called by another name, such as "bundling" and "unbundling."

So, how does this work? First of all, given that every operation in every place-value column (except the units column) contains a −1, consider what is really happening in every column. Using the 10s column values from the problem above, originally the minuend was 4 and the subtrahend was 6. This subtrahend was subtracted from 10, but also note that there is a −1 in that column and that by the commutative and associative properties, the −1 can be subtracted from the 10. So rather than 10 – 6, we have 9 – 6, or 3; and this added to the 4 gives the correct answer of 7. This can be done for the values in every column, except the unit's column because there is no −1 in that column. So, look at how the original problem would look if the 9s complement were employed:

The original problem was:

	1	2	3	4	5
−		5	1	6	9

Before applying the 9s complement, note that again there is the issue of the 10-thousands column. Either there is no value and, therefore, there is no 9s complement; or we can provide a leading zero, in which case the 9s complement is 9. Showing the problem with no value in the 10-thousands column gives:

	1	2	3	4	5
+		4	8	3	0
		7	1	7	5

Uh-oh. This answer is off by 1; the correct answer is 7176. Will all problems come out like this? If so, the correct answer can be found by always adding 1 to the answer from the 9s complement work. Before describing and demonstrating the justification for the addition of 1 to any answer, take a look at what happens if a zero is allowed in the 10-thousands column of the subtrahend. The complement of zero is 9 so the problem would become:

	1	2	3	4	5
+ 9		4	8	3	0
1 0		7	1	7	5

How is the 1 in the 100-thousands column to be handled? How about adding it to the 5 in the units column, which would solve the problem of being 1 short in that column?

Technically, it does not get "carried" from the 100-thousands column to the units column, but watch what happens arithmetically to show why the units column is to be increased by 1 in every 9s complement subtraction.

Using the original problem of 12345 – 5169, the first step is to add zero to it in the form of 10,000 – 10,000, giving:

10,000 + 12345 – 5169 – 10,000

Then replace 10,000 with 9999 + 1, giving

9999 + 1 + 12345 – 5169 – 10,000.

A little commutation gives

1 + 12345 + 9999 – 5169 – 10,000.

A little association gives

(1 + 12345) + (9999 – 5169) – 10,000.

Doing the indicated operations inside each grouping symbol gives

12346 + 4830 − 10,000.

Do the first two values look familiar? They are the original minuend plus one and the 9s complement of the subtrahend. Again, doing the indicated operations from left to right gives:

17176 − 10,000 and finally, the correct answer 7176.

In Summary

This lengthy detailed "slow-motion" description of how a child's simple error has the roots of a very sophisticated subtraction algorithm (and without borrowing or regrouping!) is well worth the effort. It shows how threads of arithmetical and mathematical concepts and operations connect the simple to the sublime. And by the way, your computer uses binary-complements subtraction, so given this kid's error, s/he may be on the way to being a programmer!

And one final note: A key element of teaching math can be seen as identifying not just errors but the origins of errors. Not all errors have the above-noted threads to more powerful conceptions, but experience has demonstrated that many of the consistent and hard-to-correct student misconceptions have roots in our presentation pedagogy; and computer-based conceptions do not help instructors or students identify the origins of the errors. The mix of language and symbols employed in math mask what is really evolving in a child's head as s/he navigates the utility of math algorithms. And once a child has chosen a particular path, and that path provides correct answers most of the time, why bother examining error patterns for clues for correcting a misconception?

If you are a math instructor, take a moment to reflect on a student's error, and consider the following: Is this an occasional error because of fatigue or distraction, or is this an ingrained error the student learned how to make?

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